Module 2A - Heat Transfer

Omar Betancourt, Payton Goodrich, Emre Mengi

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Objectives: To understand and model heat flow and solve for temperature gradients using analytical methods.

Prerequisite Knowledge: High-school physics

Prerequisite Modules: 1A - Calculus, 1D - Differential Equations

Difficulty: Easy

Summary: In this module, you will learn about the three forms of heat transfer and how to express them in an energy balance equation.

1 Theory

1.1 Conservation of Energy

As you may know, heat is a form of energy, which means that it is subject to following the first law of thermodynamics which elegantly states that energy cannot be created or destroyed, only converted from one form of energy to another or stored:

$$\Delta E_{st}^{tot} = Q - W \tag{1.1}$$

where E_{st}^{tot} is the energy stored in the system, Q is the heat transferred to the system, and W is the net work done by the system.

The first law addresses the *total* energy of a system, which consists of potential, kinetic, and internal energies. Internal energy too can be subdivided into several types of energy, such as thermal energy, chemical energy, nuclear energy, etc. For the study of heat transfer, however, we will focus specifically on thermal energy in this module. In the case where all energy transfer is thermal, Equation 1.1 can be reduced to:

$$m_i C_i \dot{\theta}_i = Q_{cond} + Q_{conv} + Q_{rad} + Q_{gen} \tag{1.2}$$

where m_i is the mass of a control volume i, C_i is the heat capacity per unit mass of the control volume, θ_i is the change in temperature for the time step, and Q is the heat transferred. Equation 1.2 is a lumped thermal model, and is valid as long as the lumped mass elements are small. This is particularly useful for numerical methods that will be discussed in later modules.

Heat transfer, Q, is the transport of thermal energy, and has units of W. Heat transfer occurs whenever there is a temperature difference in a medium or between media, and can be transferred in three different modes:

1. Conduction through a solid or stationary fluid
2. Convection from a surface to a moving fluid
3. Radiation between two surfaces

In the following subsections, you will learn about each of these methods of heat transfer.

1.2 Conduction

Conduction occurs through atomic or molecular activity. High temperatures are associated with higher molecular energies, which increases the rate that these molecules vibrate. As these molecules collide (which all molecules are constantly doing) energy is transferred from the higher energy molecules to the lower energy ones. The rate that this energy transfer occurs is known as Fourier's Law:

$$\mathbf{q}" = \frac{Q_{cond}}{A} = -\mathbb{K}\nabla\theta(x, y, z) \tag{1.3}$$

where $\mathbf{q}"$ is the heat flux and $\mathbb{K}$ is the material's thermal conductivity, and ∇ is the three-dimensional del operator. Equation 1.3 implies that heat flux is directional and normal to a surface of constant temperature, sometimes called an isotherm. For isotropic materials whose thermal conductivity does not vary in direction, $\mathbb{K}$ can be replaced by the scalar thermal conductivity coefficient k.

1.3 Convection

Convection occurs in a fluid medium, and is driven by two mechanisms. First is a transfer of random molecular motion, or diffusion, to the fluid. The second is the macroscopic fluid motion in the presence of a temperature gradient. We are particularly interested in convection that occurs between a flowing fluid medium and a surface that are at two different temperatures. At the fluid-solid surface, both a hydrodynamic and thermal boundary layer forms. These two boundary layers are coupled, but the overall heat transfer rate can be calculated:

$$\mathbf{q}" = \frac{Q_{conv}}{A} = h\left(\theta_s - \theta_\infty\right) \tag{1.4}$$

where h is the convection coefficient, which accounts for not only many fluid properties such as density, viscosity, and thermal conductivity, but also the surface geometry and flow conditions. The multiplicity of independent variables involved in the formulation of h is a complex one that is most often found empirically.

1.4 Radiation

Radiation occurs between any two finite volumes of matter at a different temperature, regardless of the medium between them. While technically all forms of matter emit thermal radiation, we will concern ourselves only with surface-to-surface radiation of solids as this is the most prevalent. Thermal radiation is the propagation of electromagnetic waves, approximately in the 0.1 μm - 100μm wavelength which varies depending on the absolute temperature of the surface. Radiation is also *directional*, meaning the intensity that a surface radiates thermal energy varies with respect to the direction.

Various types of heat fluxes contribute to radiation heat transfer, which are outlined in Table 1.4.

Radiative Fluxes

Flux	Description	Comment
Emmisive Power, E	Rate that radiation is emmited from a surface per unit area	$E = \epsilon\beta\theta_s^4$
Irradiation, I	Rate at which radiation is incident upon a surface per unit area	Irradiation can be reflected, absorbed, or transmitted, generally dependent on the transparency of the incident surface
Radiosity, J	Rate that radiation leaves a surface per unit area	For opaque surfaces, $J = E + \rho I$
Net Radiative Flux, $q"_{rad} = J - I$	Net rate of radiation leaving a surface per unit area	For opaque surfaces, $q"_{rad} = \epsilon\beta\theta_s^4 - \alpha I$

The upper limit of the emissive power, E, of a body is defined by Steffan-Boltzman law:

$$E = \beta\theta_s^4 \tag{1.5}$$

where β is the Steffan-Boltzman constant, $\beta = 5.67 \cdot 10^{-8} \frac{W}{m^2 \cdot K}$, and θ_s is the temperature of the surface. Such a surface is an ideal radiator, also known as a blackbody. Non-ideal transmitters include an emissivity

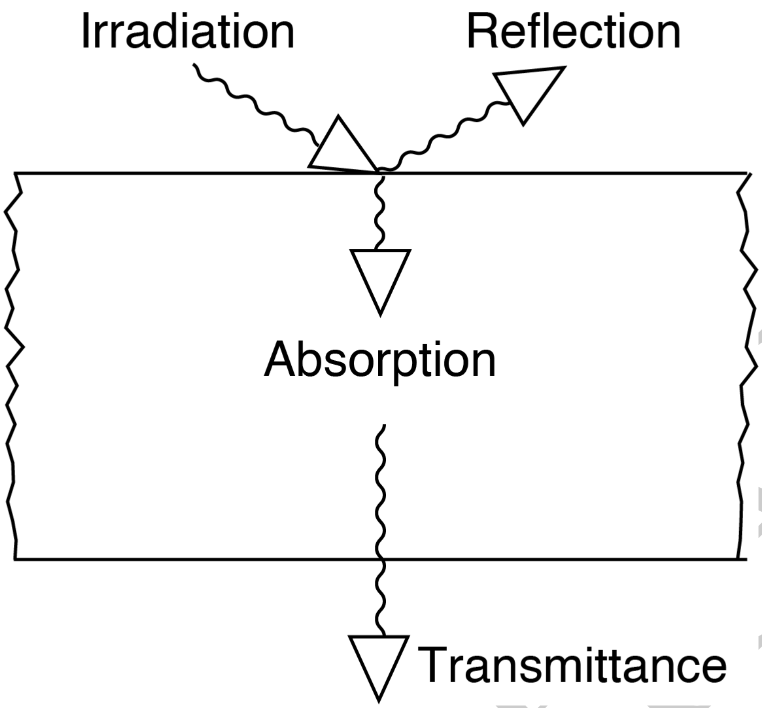

Figure 1.1: Visualization of the net irradiation fluxes at a surface. The efficiency that a body absorbs irradiation is given by the absorption coefficient α.

constant, $0 \leq \epsilon \leq 1$, which is a measure of how efficiently the surface emits radiation compared to a blackbody.

$$E = \epsilon\beta\theta_s^4 \tag{1.6}$$

Combining this with the amount of radiative flux that is transmitted through the receiving body and the amount that leaves the receiving body, we have:

$$\mathbf{q}" = \frac{Q_{rad}}{A} = \epsilon\beta\theta_s^4 - \alpha I \tag{1.7}$$

where $0 \leq \alpha \leq 1$ is an absorption coefficient. The more transparent a surface is, the smaller the value of α is as more light is transmitted through the surface. Similarly, the more reflective a surface is will cause the heat be reflected to the surroundings. The absorption coefficient accounts for all of these inefficiencies, and is often found experimentally or estimated.

A common special case that occurs is when there is radiative exchange between one surface and a much larger, isothermal surface that completely surrounds the smaller surface. An example for this would be a piece of food placed in an oven, or the same piece of food cooling after it is taken out of the oven. In this case, Equation 1.7 becomes:

$$\mathbf{q}" = \frac{Q_{rad}}{A} = \epsilon\beta\left(\theta_s^4 - \theta_\infty^4\right) \tag{1.8}$$

1.5 The Heat Diffusion Equation

It is often useful to know the temperature field in a medium that occurs when thermal conditions are imposed on its boundaries. For solids, knowledge of the temperature distribution can determine structural integrity, compatibility with coatings and adhesives, or required cooking time for safe eating. To find a temperature distribution, we define a differential control volume, identify relevant heat transfer processes, and apply the appropriate heat rate equations.

Following Equation 1.2, we first define an infinitesimal control volume dx, dy, dz as shown in Figure 1.2. Next is to consider the heat transfer processes. As a reminder, recall that in this module we are assuming that mechanical and potential energy processes are negligible, though you may need to account for them in future applications. If there are temperature gradients, conduction will occur across each of our control surfaces. Likewise, it needs to be determined if convective or radiative heat transfer is applicable. Finally, there may also be an energy source term within the medium that generates thermal energy, E_g. Assuming all of these terms exist, the energy balance across the control volume and all of its surfaces becomes:

$$Q_{in} + Q_{gen} - Q_{out} = Q_{st} \tag{1.9}$$

Because the heat equations account for both heat inflow and outflow, we can substitute Equations 1.3, 1.4, and 1.7 into Equation 1.9 to obtain:

$$-\mathbb{K}_i A_i \nabla\theta_i(x,y,z) - h_i A_i\left(\theta_i - \theta_\infty\right) - \epsilon\beta A_i\left(\theta_i^4 - \theta_\infty^4\right) + \alpha_i A_i I + Q_{gen} = m_i C_i \frac{\partial \theta_i}{\partial t} \tag{1.10}$$

where q_{gen} is the heat generated per unit volume. This is the general form in cartesian coordinates and can be further simplified further, for example if the material is isotropic or the system is in steady-state conditions.

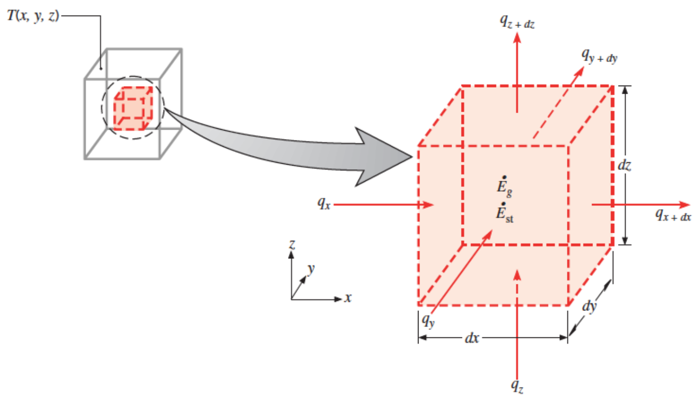

Figure 1.2: Differential control volume dx, dy, dz.

2 Example

Consider a 25-mm-thick polystyrene food container with interior dimensions of 0.8 m × 0.5 m × 0.5 m. The container has an inner surface temperature of approximately 2°C that is maintained by an ice-water mixture and an outer surface temperature of 25 °C is maintained by the ambient temperature of the room that it is in. What is the heat flux through the container walls? Assuming negligible heat gain through the 0.8 m × 0.5 m base of the cooler, what is the total heat load for the prescribed conditions?

Using the thermal conductivity of polystyrene $k = 0.023\frac{W}{m \cdot K}$, we can calculate the thermal conductivity as:

$$q^{"}_{cond} = k\frac{\theta_2 - \theta_1}{L} = \frac{0.023 \text{ W/m} \cdot \text{K}(25-2)°\text{C}}{0.025 \text{ m}} = 21.16 \text{ W/m}^2 \tag{2.1}$$

Since the heat flux is uniform over each of the walls of the container (except for the bottom), we have:

$$\begin{aligned} \text{Q}_{\text{cond}} &= \text{q}''_{cond} \times \text{A}_{\text{total}} = \text{q}''_{cond}\left[\text{H}\left(2\ \text{W}_1 + 2\ \text{W}_2\right) + \text{W}_1 \times \text{W}_2\right] \\ \text{Q}_{\text{cond}} &= 21.16 \text{ W/m}^2[0.5 \text{ m}(1.6 \text{ m} + 1.0 \text{ m}) + (0.8 \text{ m} \times 0.5 \text{ m})] = 35.97 \text{ W} \end{aligned} \tag{2.2}$$

3 Assignment

Microwaves operate by rapidly reversing the alignment of water molecules in food, resulting in a volumetric energy generation that increases the temperature of the food, effectively cooking it. When food is frozen, however, the water molecules do not readily reverse their alignment and the volumetric energy generation is much less.

Consider a 0.1-kg ($\approx \frac{1}{4}$-lb.) disc-shaped patty of beef that is initially at -3°C placed in the center of a microwave oven whose walls are 25°C and $h = 15\frac{W}{m^2 \cdot K}$. How long will it take for the patty to thaw if it absorbs 5% of the microwaves 1 kW power?

After all of the ice in the beef patty has melted, how long will it take for the beef to reach 65°C if it now absorbs 95% of the microwaves power?

You may have noticed that when you use a microwave to heat frozen food, the middle can remain frozen even when the outside reaches the temperature of molten lava. Why does this 'Hot Pocket'[1] effect happen? How might you prevent this from happening by adjusting the microwave power and microwave time settings?
In your analysis, use the thermal properties for ice and water which are provided in the variable glossary below. Write your answers to the questions in the following order:

1. Time for the beef to reach 0°C
2. Time for the beef to be heated from 0°C to 65°C
3. Qualitative answer to why some food remains frozen in the middle after microwaving, and strategies to avoid this phenomenon.

VARIABLE GLOSSARY

Symbol	Type	Units	Value	Description
m	Scalar	kg	0.1	Mass
ρ	Scalar	$\frac{kg}{m^3}$	1000	Density
h	Scalar	$\frac{W}{m^2 \cdot K}$	15	Convection coefficient
C_{ice}	Scalar	$\frac{J}{kg \cdot K}$	2040	Heat capacity of ice per unit mass
C_{water}	Scalar	$\frac{J}{kg \cdot K}$	4179	Heat capacity of water per unit mass
Q_{max}	Scalar	W	1000	Power
η_1	Scalar	unitless	0.05	Heat Generation Efficiency - Frozen
η_2	Scalar	unitless	0.95	Heat Generation Efficiency - Thawed
θ_0	Scalar	$^\circ C$	-3	Initial Temperature
θ_∞	Scalar	$^\circ C$	25	Temperature of Surroundings
θ_f	Scalar	$^\circ C$	65	Desired Temperature

4 Solution

Time for the beef to reach 0°C
First apply conservation of energy:

$$\begin{aligned} Q_{in} + Q_{gen} &= Q_{st} \\ hA_s\left(\theta_\infty - \theta\right) + Q_{gen} &= mC\frac{\partial\theta}{\partial t} \end{aligned} \tag{4.1}$$

With initial condition $\theta(t = 0) = \theta_0$. Define:

$$X = \theta - \theta_\infty - \frac{Q_{gen}}{hA_s} \tag{4.2}$$

[1] Here we refer to an enclosed volume with high temperature. Any similarity to trademarked products is purely coincidental.

and substitute into Equation 4.1:

$$\frac{\partial X}{\partial t} = -\frac{hA_s}{mC}\theta \tag{4.3}$$

The variables are then separated and integrated:

$$\begin{aligned} &\int_{X(0)}^{X(\mathrm{t})} \frac{\partial X}{X} = -\frac{\mathrm{hA_s}}{\mathrm{mC}} \int_0^t \mathrm{dt} \\ &\ln\left[\frac{X(\mathrm{t})}{X(0)}\right] = -\frac{\mathrm{hA_s t}}{\mathrm{mC}} \\ &\ln\left[\frac{\theta-\theta_\infty-\mathrm{Q_{gen}/hA_s}}{\theta_0-\theta_\infty-\mathrm{Q_{gen}/hA_s}}\right] = -\frac{\mathrm{hA_s t}}{\mathrm{mC}} \end{aligned} \tag{4.4}$$

The area of the beef surface is not given in the problem definition but can be estimated from the mass and density. Assuming a patty thickness, $z = 20$-cm, we can calculate the patty radius, r:

$$\begin{aligned} &m = \rho V \\ &V = \frac{m}{\rho} = \pi r^2 z \\ &r = \sqrt{\frac{m}{\rho\pi z}} \approx 13cm \end{aligned} \tag{4.5}$$

The surface area of a cylindrical patty is given by:

$$A_s = 2\pi r z + 2\pi r^2 \approx 0.16m^2 \tag{4.6}$$

Substituting A_s and the rest of the given variables into Equation 4.4 gives:

$$\ln\left[\frac{0°\mathrm{C} - 25°\mathrm{C} - 50/(15 \cdot 0.16)}{-3°\mathrm{C} - 25°\mathrm{C} - 50/(15 \cdot 0.16)}\right] = -\frac{15 \cdot 0.16}{0.1 \cdot 2040}\mathrm{t} \tag{4.7}$$

which gives t = 447s, or about seven and a half minutes.

Time for the beef to be heated from 0°C to 65°C

After all of the ice has melted to water, we can again use Equation 4.4 with values of $\theta_0 = 0°C$, $\theta = \theta_f = 65°C$, $C = 4179\frac{J}{kg \cdot K}$, and $\eta_2 = 0.95$:

$$\ln\left[\frac{65°\mathrm{C} - 25°\mathrm{C} - 950/(15 \cdot 0.16)}{0°\mathrm{C} - 25°\mathrm{C} - 950/(15 \cdot 0.16)}\right] = -\frac{15 \cdot 0.16}{0.1 \cdot 4179}\mathrm{t} \tag{4.8}$$

which gives t = 46s. Thawed food will rise in temperature at a much faster rate than frozen food.

Qualitative answer to why some food remains frozen in the middle after microwaving, and strategies to avoid this phenomenon

Liquid water absorbs microwave energy much better than frozen water. If microwave irradiation is non-uniform, then the subsequent temperature rise will also be non-uniform: thawed regions of the food will absorb more energy than frozen regions and have a much faster rise in temperature. If the food has a low thermal conductivity, there may not be enough time for thermal conduction to make temperature more uniform. Using lower power settings and longer microwave times will allow thermal conduction to equalize the temperature throughout the food dish and cook more evenly.

The solutions to parts one and two are encoded in Matlab below:

```
%% Variables

m = 0.1;
k = 0.45;
h = 15;
rho = 1000;
C1 = 2040;
C2 = 4179;
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Q_max = 1000;
theta_0 = -3;
theta_inf = 25;
theta_1 = 0;
theta_2 = 65;
eff1 = 0.05;
eff2 = 0.95;

%% Prob 1

V = m / rho;
z = 20;
r = sqrt(m/(rho*pi*z));
A_s = 2*pi*r*z + 2*pi*(r^2);

t1 = log((theta_1 - theta_inf - ((Q_max*eff1)/(h*A_s)))/...
    (theta_0 - theta_inf - ((Q_max/eff1)/(h*A_s))))/((h*A_s)/(-m*C1));

%% Prob 2

t2 = log((theta_2 - theta_inf - ((Q_max*eff2)/(h*A_s)))/...
    (theta_1 - theta_inf - ((Q_max/eff2)/(h*A_s))))/((h*A_s)/(-m*C2));
```

5 References

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