Module 1B - Linear Algebra

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Contents

Objectives: Review key linear algebra concepts as they pertain to subsequent modules.

Prerequisite Knowledge: Algebra

Prerequisite Modules: 1A - Calculus

Difficulty: Easy

Summary: This module reviews key linear algebra concepts that will be used future modules.

1 Theory

If we consider the second order tensor $\boldsymbol{A}$ with its matrix representation

$$[\boldsymbol{A}] \overset{\text{def}}{=} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}. \tag{1.1}$$

The matrix $[\boldsymbol{A}]$ is said to be symmetric if $[\boldsymbol{A}] = [\boldsymbol{A}]^T$ and skew-symmetric if $[\boldsymbol{A}] = -[\boldsymbol{A}]^T$. A first order contraction (inner product) of two matrices is defined by

$$\boldsymbol{A} \cdot \boldsymbol{B} = [\boldsymbol{A}][\boldsymbol{B}] \text{ which has components of } \sum_{j=1}^{N} A_{ij} B_{jk} = C_{ik}, \tag{1.2}$$

where it is clear that the range of the inner index j must be the same for $[\boldsymbol{A}]$ and $[\boldsymbol{B}]$. The second order inner product of two matrices is

$$\boldsymbol{A} : \boldsymbol{B} = A_{ij} B_{ij} = \text{tr}\left([\boldsymbol{A}]^T [\boldsymbol{B}]\right). \tag{1.3}$$

Some properties of the determinant are (where, for example, $[\boldsymbol{A}]$ is a 3×3 matrix):

- $det[\boldsymbol{A}] = A_{11}\left(A_{22}A_{33} - A_{32}A_{23}\right) - A_{12}\left(A_{21}A_{33} - A_{31}A_{23}\right) + A_{13}\left(A_{21}A_{32} - A_{31}A_{22}\right)$
- $det[\boldsymbol{I}] = 1$, $\det \alpha[\boldsymbol{A}] = \alpha^3 det[\boldsymbol{A}]$, where α is a scalar,
- $det[\boldsymbol{A}][\boldsymbol{B}] = det[\boldsymbol{A}]det[\boldsymbol{B}], det[\boldsymbol{A}]^T = det[\boldsymbol{A}]$ and $det[\boldsymbol{A}]^{-1} = \frac{1}{det[\boldsymbol{A}]}$

An important use of the determinant is in forming the inverse by

$$[\boldsymbol{A}]^{-1} = \frac{\text{adj}[\boldsymbol{A}]}{\det[\boldsymbol{A}]}, \quad \text{adj}[\boldsymbol{A}] \overset{\text{def}}{=} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T, \tag{1.4}$$

where the so-called co-factors are

$$\boxed{\begin{array}{ll} C_{11} = A_{22}A_{33} - A_{32}A_{23} & C_{12} = -\left(A_{21}A_{33} - A_{31}A_{23}\right) \\ C_{13} = A_{21}A_{32} - A_{31}A_{22} & C_{21} = -\left(A_{12}A_{33} - A_{32}A_{13}\right) \\ C_{22} = A_{11}A_{33} - A_{31}A_{13} & C_{23} = -\left(A_{11}A_{32} - A_{31}A_{12}\right) \\ C_{31} = A_{12}A_{23} - A_{22}A_{13} & C_{32} = -\left(A_{11}A_{23} - A_{21}A_{13}\right) \\ C_{33} = A_{11}A_{22} - A_{21}A_{12} & \end{array}} \tag{1.5}$$

The rule of transposes for two $n \times n$ matrices

$$([\boldsymbol{A}][\boldsymbol{B}])^T = [\boldsymbol{B}]^T [\boldsymbol{A}]^T \tag{1.6}$$

The rule of inverses for two invertible $n \times n$ matrices is

$$([\boldsymbol{A}][\boldsymbol{B}])^{-1} = [\boldsymbol{B}]^{-1}[\boldsymbol{A}]^{-1} \quad \text{and} \quad [\boldsymbol{A}]^{-1}[\boldsymbol{A}] = [\boldsymbol{A}][\boldsymbol{A}]^{-1} = [\boldsymbol{I}] \tag{1.7}$$

where $[\boldsymbol{I}]$ is the identity matrix. Clearly, $[\boldsymbol{A}]^{-1}$ exists only when $det[\boldsymbol{A}] \neq 0$. The mathematical definitions of an eigenvalue, a scalar denoted Λ and eigenvector, a vector denoted $\mathcal{E}$, of a matrix $[\boldsymbol{A}]$ are

$$[\boldsymbol{A}]\{\mathcal{E}\} = \Lambda\{\mathcal{E}\} \tag{1.8}$$

We note that for any given tensor $\boldsymbol{A}$ of order 2 (a 3×3 matrix), if we set the determinant $det[\boldsymbol{A} - \Lambda \boldsymbol{I}] = 0$, it can be shown that the so-called "characteristic" polynomial is

$$det[\boldsymbol{A} - \Lambda \boldsymbol{I}] = -\Lambda^3 + I_A \Lambda^2 - II_A \Lambda + III_A = 0 \tag{1.9}$$

where

$$\boxed{\begin{array}{l} I_A = tr(\boldsymbol{A}) = \Lambda_1 + \Lambda_2 + \Lambda_3 \\ II_A = \frac{1}{2}\left((tr(\boldsymbol{A}))^2 - tr\left(\boldsymbol{A}^2\right)\right) = \Lambda_1\Lambda_2 + \Lambda_2\Lambda_3 + \Lambda_1\Lambda_3 \\ III_A = det(\boldsymbol{A}) = \frac{1}{6}\left((tr\boldsymbol{A})^3 + 2tr\boldsymbol{A}^3 - 3\left(tr\boldsymbol{A}^2\right)(tr\boldsymbol{A})\right) = \Lambda_1\Lambda_2\Lambda_3 \end{array}} \tag{1.10}$$

Since I_A, II_A and III_A, can be written in terms of $tr\boldsymbol{A}$, which is invariant under frame rotation, they too are invariant under frame rotation. The main properties to remember about eigenvalues and eigenvectors are:

1. If $[\boldsymbol{A}](n \times n)$ has n linearly independent eigenvectors then it is diagonalizable by a matrix formed by columns of the eigenvectors, for example for a 3×3 matrix

$$\begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{bmatrix} = \begin{bmatrix} \mathcal{E}_1^{(1)} \mathcal{E}_1^{(2)} \mathcal{E}_1^{(3)} \\ \mathcal{E}_2^{(1)} \mathcal{E}_2^{(2)} \mathcal{E}_2^{(3)} \\ \mathcal{E}_3^{(1)} \mathcal{E}_3^{(2)} \mathcal{E}_3^{(3)} \end{bmatrix}^{-1} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \mathcal{E}_1^{(1)} \mathcal{E}_1^{(2)} \mathcal{E}_1^{(3)} \\ \mathcal{E}_2^{(1)} \mathcal{E}_2^{(2)} \mathcal{E}_2^{(3)} \\ \mathcal{E}_3^{(1)} \mathcal{E}_3^{(2)} \mathcal{E}_3^{(3)} \end{bmatrix} \tag{1.11}$$

2. If $[\boldsymbol{A}](n \times n)$ has n distinct eigenvalues then the eigenvectors are linearly independent

3. If $[\boldsymbol{A}](n \times n)$ is symmetric then its eigenvalues are real. If the eigenvalues are distinct, then the eigenvectors are orthogonal.

A quadratic form is such that

$$\{\boldsymbol{x}\}^T[\boldsymbol{A}]\{\boldsymbol{x}\} \stackrel{\text{def}}{=} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{1.12}$$

A matrix $[\boldsymbol{A}]$ is said to be positive definite if the quadratic form is positive for all nonzero vectors $\boldsymbol{x}$. Clearly, from equation 2.19 a positive definite matrix must have positive eigenvalues.

To perform a coordinate transform for a 3×3 matrix $[\boldsymbol{A}]$ from one Cartesian coordinate system to another (denoted with a $(\hat{\cdot})$), we apply a transformation matrix $[\boldsymbol{Q}]$:

$$[\hat{\boldsymbol{A}}] = [\boldsymbol{Q}][\boldsymbol{A}][\boldsymbol{Q}]^{-1}$$

If $\boldsymbol{Q}$ is an orthogonal matrix, then $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$ (denoted "unitary").

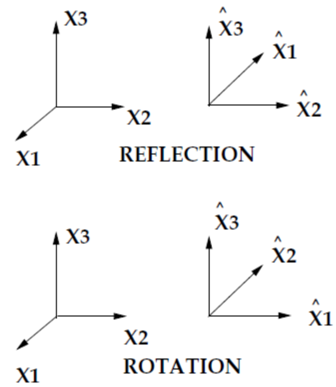

Figure 1.1: Top: reflection with respect to the $x_2 - x_3$ plane. Bottom: rotation with respect to the x_3 axis.

The standard axes rotators are Figure 1.1, about the x1 axis

$$Rot\,(x_1) \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}, \tag{1.13}$$

about the x_2 axis

$$Rot\,(x_2) \stackrel{\text{def}}{=} \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}, \tag{1.14}$$

about the x_3 axis

$$Rot\,(x_3) \stackrel{\text{def}}{=} \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.15}$$

The standard axes reflectors are, with respect to the $x_2 - x_3$ plane

$$Ref\,(x_1) \stackrel{\text{def}}{=} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.16}$$

with respect to the $x_1 - x_3$ plane

$$Ref\,(x_2) \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.17}$$

with respect to the $x_1 - x_2$ plane

$$Ref\,(x_3) \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{1.18}$$

2 Example

Two example problems are given below:

1. We can compute the determinant of a matrix as shown below:

   $$A = LU, \text{ where } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

   Calculating the determinant of the lower triangle matrix and the upper triangle matrix and multiplying the results would give us the determinant of A.

   $$det(A) = det(L)det(U) = (2)(7) = 14$$

2. To find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -10 & 4 \\ -18 & 12 \end{bmatrix}$.

   $$det(A - \lambda I) = \begin{vmatrix} -10 - \lambda & 4 \\ -18 & 12 - \lambda \end{vmatrix} = 0$$

   $$\lambda^2 - 2\lambda - 48 = 0$$

   $$\lambda_1 = 8, \quad \lambda_2 = -6$$

   $$A \cdot v_1 = \lambda_1 I \cdot v_1 \rightarrow (A - \lambda_1 I)v_1 = 0$$

   $$A - \lambda_1 I = \begin{bmatrix} -10 - 8 & 4 \\ -18 & 12 - (8) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -18 & 4 \\ -18 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   $v_1 = [0, 0]^T$ is a trivial solution. Looking at nontrivial solutions: choosing the y-component of v_1 to be 1, the first eigenvector is:

   $$v_1 = [2/9, 1]^T$$

   With the same steps, the second eigenvector is:

   $$v_2 = [1, 1]^T$$

3 Assignment

LINEAR ALGEBRA REVIEW (100 POINTS)

In this assignment, you will be reviewing the basics concepts of linear algebra. *The typical report format will not be required for this assignment, and unless otherwise specified, either hand-written or typed information is acceptable.* If you use Matlab or another language for simplifying answers or evaluating specific numbers, clearly set up the equations before showing your numerical results. Correct answers without supporting work will be penalized, and incorrect answers without supporting work will receive zero credit.

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1. Find the solution to the system of equations (10 points)

$$\begin{aligned} 3w + 2x - 4y &= 0 \\ -2w - 2x + y &= 5 \\ w + x + y &= 1. \end{aligned}$$

2. Compute the determinants for the following matrices (20 points)

   - (a) $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & -5 & 0 \\ 0 & 3 & 7 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1/2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$
   - (b) $C = \begin{bmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{bmatrix}$, where x, y, z are nonzero and distinct.

3. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 2 \\ -4 & 6 \end{bmatrix}$. (20 points)

4. Find values of t that result in no solutions to the following system of linear equations. (20 points)

$$\begin{aligned} 2x_1 + 2tx_2 + 2tx_3 &= 2t, \\ 7x_1 + 7tx_2 + 7t^3x_3 &= 7. \end{aligned}$$

   How many solutions are there when $t = 1$?

5. Show that the following vectors are mutually orthogonal. (10 points)

$$\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ -6 \\ 6 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -6 \\ 3 \\ 6 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}$$

6. Evaluate whether matrix A is orthogonal. (20 points)

$$A = \begin{bmatrix} 2/7 & -4/7 & 4/7 \\ -4/7 & 2/7 & 4/7 \\ 4/7 & 4/7 & 2/7 \end{bmatrix}$$

4 Solution

1.
$$w = 12$$
$$x = -13.3$$
$$z = 2.3$$

2. • (a)
$$det(A) = det(L)det(U) = (-35)(6) = -210$$

   • (b)
$$det(C) = x(y^2z^3 - y^3z^2) - x^2(yz^3 - y^3z) + x^3(yz^2 - y^2z) = -x^3y^2z + x^3yz^2 + x^2y^3z - x^2yz^3 - xy^3z^2 + xy^2z^3$$

3.
$$det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 2 \\ -4 & 6-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$\lambda_1 = 4, \quad \lambda_2 = 2$$

$$A \cdot v_1 = \lambda_1 I \cdot v_1 \to (A - \lambda_1 I)v_1 = 0$$

$$v_1 = [-0.707, 0.707]^T, \quad v_2 = [-0.447, -0.894]^T$$

4. If we write
$$\begin{bmatrix} 2 & 2t & 2t & | & 2t \\ 7 & 7t & 7t^3 & | & 7 \end{bmatrix}$$

After Gaussian elimination:
$$\begin{bmatrix} 1 & t & t & | & 1 \\ 0 & 0 & t^3 - t & | & 1-t \end{bmatrix}$$

For the system to have no solution:

$$t^3 - t = 0 \quad \text{while} \quad 1 - t \neq 0$$

Then $t = -1, 0$.

When $t = 1$, the system becomes:
$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

which gives infinite solutions.

5.

$$v_1 \cdot v_2 = 3 \cdot -6 - 6 \cdot 6 + 6 \cdot 6 = 0$$

$$v_2 \cdot v_3 = -6 \cdot 6 + 3 \cdot 6 + 6 \cdot 3 = 0$$

$$v_1 \cdot v_3 = 3 \cdot 6 - 6 \cdot 6 + 6 \cdot 3 = 0$$

6.

$$A^T = \begin{bmatrix} 2/7 & -4/7 & 4/7 \\ -4/7 & 2/7 & 4/7 \\ 4/7 & 4/7 & 2/7 \end{bmatrix}$$

$$A^T A = AA^T = \begin{bmatrix} 0.735 & 0 & 0 \\ 0 & 0.735 & 0 \\ 0 & 0 & 0.735 \end{bmatrix} \neq \mathbf{I}$$

This matrix is not orthogonal.

5 References

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