Module 2C - Particle Dynamics

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Objectives:
The objective of this module is to learn how to analytically solve for a particles trajectory through Newton's second law of motion.

Prerequisite Knowledge: High-school physics

Prerequisite Modules: 1A - Calculus, 1D - Differential Equations
Difficulty: Easy

Summary: In this module, we will go over solving particle kinematics using Newton's second law. Example problems and assignment will look into tracking the trajectory of a particle undergoing gravitational and drag forces.

1 Theory

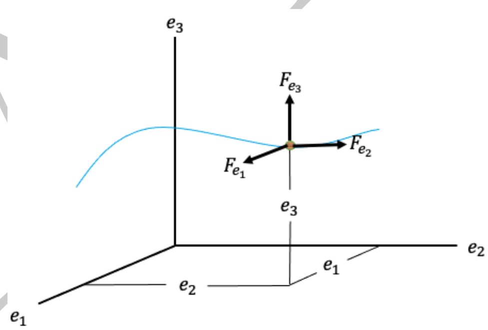
Kinetics is the study of the relation between forces and the acceleration they cause. This relation is based on Newton's second law of motion. For an arbitrary i-th particle within a system is expressed mathematically as

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{\Psi}_i^{\text{tot}}. \tag{1.1}$$

The resultant force, $\boldsymbol{\Psi}_i^{\text{tot}}$, represents, for example, the effect of gravitational, electrical, magnetic, or contact forces between the ith particle and adjacent bodies or particles not included within the system.

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{\Psi}_i^{\text{tot}}(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_{N_m}) = \boldsymbol{F}_i^{\text{grav}} + \boldsymbol{F}_i^{\text{elec}} + \boldsymbol{F}_i^{\text{drag}} + other\ external\ forces$$

This equation of motion states that the unbalanced force on a particle causes it to accelerate. Hence, the sum of the external forces action on a particle is equal to the mass of that particle times its acceleration. When a particle moves relative to a fixed Cartesian frame of reference, the forces acting on the particle can be express in terms of their $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ components.



As well as for its position, acceleration and velocity:

$$\boldsymbol{r} = r_1 \boldsymbol{e}_1 + r_2 \boldsymbol{e}_2 + r_3 \boldsymbol{e}_3 \tag{1.2}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}_1 \boldsymbol{e}_1 + \dot{r}_2 \boldsymbol{e}_2 + \dot{r}_3 \boldsymbol{e}_3 \tag{1.3}$$

$$\boldsymbol{a} = \ddot{\boldsymbol{r}} = \ddot{r}_1 \boldsymbol{e}_1 + \ddot{r}_2 \boldsymbol{e}_2 + \ddot{r}_3 \boldsymbol{e}_3 \tag{1.4}$$

For the following example and the assignment you will be tasked to track a trajectory of a particle undergoing multiple external forces.

2 Example

Consider a particle with a radius R_i. The mas is given by $m_i = \rho_i \frac{4}{3}\pi R_i^3$, where ρ_i is the density of the particle.
We assume that the particles are *quite small* and that the amount of rotation, if any, contributes negligibly to the overall trajectory of the particle. The equation of motion for this particle in the system is

$$m_i \dot{\boldsymbol{v}}_i = \boldsymbol{\Psi}_i^{grav} + \boldsymbol{\Psi}_i^{drag}, \tag{2.1}$$

with initial velocity $\boldsymbol{v}_i(0)$ and initial position $\boldsymbol{r}_i(0)$. The gravitational force is $\boldsymbol{\Psi}_i^{grav} = m_i \boldsymbol{g}$, where $\boldsymbol{g} = (g_x, g_y, g_z) = (0, 0, -9.81)\, m/s^2$.
For the drag, in order to gain insight, initially, we will discuss the closely related, analytically tractable, Stokesian model next.

2.1 Analysis of particle velocity

For a (low Reynolds number) Stokesian model, the differential equation for each particle is (Eq. 2.2)

$$m_i \frac{d\boldsymbol{v}_i}{dt} = m_i \boldsymbol{g} + c_i(\boldsymbol{v}^f - \boldsymbol{v}_i) \tag{2.2}$$

where $c_i = \mu_f 6\pi R_i$, where μ_f is the viscosity of the surrounding fluid (air) and the local Reynolds number for a particle is $Re \stackrel{\text{def}}{=} \frac{2R_i\rho_a||\boldsymbol{v}^f - \boldsymbol{v}_i||}{\mu_f}$ and μ_f is the fluid viscosity. This can be written in normalized form as

$$\frac{d\boldsymbol{v}_i}{dt} + \underbrace{\frac{c_i}{m_i}}_{a_i} \boldsymbol{v}_i = \underbrace{\boldsymbol{g} + \frac{c_i}{m_i}\boldsymbol{v}^f}_{\boldsymbol{b}_i}. \tag{2.3}$$

This can be solved analytically to yield, for example in the y direction

$$v_{iy}(t) = \underbrace{(v_{iyo} - \frac{b_{iy}}{a_{iy}})}_{A_{iy}} e^{-\frac{c_i}{m_i}t} + \underbrace{\frac{b_{iy}}{a_{iy}}}_{B_{iy}}, \tag{2.4}$$

- $a_{iy} = \frac{c_i}{m_i} = \frac{9\mu_f}{2\rho_i R_i^2}$,
- $b_{iy} = g_y + \frac{c_i}{m_i} = g_y + \frac{9\mu_f}{2\rho_i R_i^2}$,
- $A_{iy} = v_{iyo} - (g_y \frac{2\rho_i R_i^2}{9\mu_f} + v_y^f)$,
- $B_{iy} = (g_y \frac{2\rho R_i^2}{9\mu_f} + v_{iy}^f)$,

where the same holds for the x and z directions. The trends are

- As $t \to \infty$

$$v_{iy}(t = \infty) \to \frac{2 g_y \rho_i R_i^2}{9\mu_f} + v_y^f, \tag{2.5}$$

- As $R_i \to 0$

$$v_{iy}(t = \infty) \to v_y^f. \tag{2.6}$$

- The decay rate is controlled by $\frac{c_i}{m_i} = \frac{9\mu_f}{2\rho_i R_i^2}$, indicating that small particles attain ambient velocities extremely quickly.

Some special cases:

- With no gravity:

$$v_{iy}(t) = (v_{iyo} - v_y^f)e^{-\frac{c_i}{m_i}t} + v_y^f. \tag{2.7}$$

- With no damping:

$$\frac{dv_{iy}}{dt} = g_y \Rightarrow v_{iy}(t) = v_{iyo} + g_y t. \tag{2.8}$$

Again, we note that the equations are virtually the same for the x and y directions, with the direction of gravity and fluid flow being the main differentiators.

2.2 Analysis of particle positions

From the fundamental equation, relating the position $\boldsymbol{r}_i$ to the velocity

$$\frac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i, \tag{2.9}$$

we can write for the y direction

$$\frac{dr_{iy}}{dt} = v_{iy} = A_{iy}e^{-\frac{c_i}{m_i}t} + B_{iy} \tag{2.10}$$

with the same being written for the x and y directions. Integrating and applying the initial conditions yields

$$r_{iy}(t) = r_{iyo} + \frac{m_i}{c_i}A_{iy}(1 - e^{-\frac{c_i}{m_i}t}) + B_{iy}t. \tag{2.11}$$

If $g_y = 0$ and $v_y^f = 0$, then

$$r_{iy}(t) = r_{iyo} + v_{iyo}2\rho_i\frac{R_i^2}{9\mu_f}(1 - e^{-\frac{9\mu_f}{2\rho_i R_i^2}t}). \tag{2.12}$$

As $t \to \infty$

$$r_{iy}(\infty) = r_{iyo} + v_{iyo}2\rho_i\frac{R_i^2}{9\mu_f}. \tag{2.13}$$

As $R_i \to 0$, the travel distance is dramatically shorter. The converse is true, larger particles travel farther.

2.3 Settling (airborne) time

The settling, steady-state velocity can be obtained directly from

$$\frac{d\boldsymbol{v}_i}{dt} + a_i\boldsymbol{v}_i = \boldsymbol{b}_i, \tag{2.14}$$

by setting $\frac{d\boldsymbol{v}_i}{dt} = \boldsymbol{0}$, one can immediately solve for the steady-state velocity

$$\boldsymbol{v}_i(\infty) = \frac{\boldsymbol{b}_i}{a_i} = \frac{2\rho_i R_i^2}{9\mu_f}\boldsymbol{g} + \boldsymbol{v}^f. \tag{2.15}$$

The trends are

- As $R_i \to 0$, then $\boldsymbol{v}_i(\infty) \to \boldsymbol{v}^f$,
- As $\boldsymbol{v}_i^f \to \boldsymbol{0}$, then $\boldsymbol{v}_i(\infty) \to \frac{2\rho R_i^2}{9\mu_f}\boldsymbol{g}$.

In summary

- Large particles travel far and settle quickly and
- Small particles do not travel far and settle slowly.

Remark: The ratio of the Stokesian drag force to gravity is

$$\frac{||\boldsymbol{\Psi}^{drag,Stokesian}||}{||\boldsymbol{\Psi}^{grav}||} = \frac{9\mu_f||\boldsymbol{v}^f - \boldsymbol{v}_i||}{2\rho_i R_i^2 g}, \tag{2.16}$$

which indicates that for very small particles, drag will dominate the settling process and for larger particles, gravity will dominate.

3 Assignment

Consider a simplified drag model for computing the trajectory of a particle.

$$m_i \frac{d\boldsymbol{v}_i}{dt} = m_i\boldsymbol{g} + c_i(\boldsymbol{v}^f - \boldsymbol{v}_i)$$

where $c_i = \mu_f 6\pi R_i$, where μ_f is the viscosity of the surrounding fluid (air). $\boldsymbol{v}_i(t=0) = \boldsymbol{v}_{i,0}$

(a) (10 points) Analytically derive an expression for the velocity in the z direction

(b) (5 points) What are the trends for velocity in the z direction as $R_i \to 0$ and $t \to \infty$?

(c) (10 points) Analytically derive an expression for the position in the z direction.

(d) (5 points) What are the trends for position in the z direction as $R_i \to 0$ and $t \to \infty$?

4 Solution

A: z-direction: For this first order differential equation we first rewrite it into the form of $\frac{dv}{dt} + p(t)v = g(t)$

$$m_i \frac{dv_{i,z}}{dt} = m_i g_z + c_i(v_z^f - v_{i,z}) \tag{4.1}$$

$$\frac{dv_{i,z}}{dt} = g_z + \frac{c_i}{m_i}(v_z^f - v_{i,z}) \tag{4.2}$$

$$\frac{dv_{i,z}}{dt} + \frac{c_i}{m_i}v_{i,z} = g_z + \frac{c_i}{m_i}v_z^f \tag{4.3}$$

Here we see $\frac{c_i}{m_i} = p(t)$ and $g_z + \frac{c_i}{m_i}v_z^f = g(t)$. We will now define our integrating factor $\mu(t)$.

$$\mu(t) = e^{\int p(t)dt} = e^{\frac{c_i}{m_i}t}$$

Now multiply all the terms in Eq. 4.3 by the integrating factor and so some simplification.

$$e^{\frac{c_i}{m_i}t}\frac{dv_{i,z}}{dt} + e^{\frac{c_i}{m_i}t}\frac{c_i}{m_i}v_{i,z} = e^{\frac{c_i}{m_i}t}(g_z + \frac{c_i}{m_i}v_z^f) \tag{4.4}$$

By the product rule of calculus this turns into

$$\int (e^{\frac{c_i}{m_i}t} v_{i,z})' = e^{\frac{c_i}{m_i}t}(g_z + \frac{c_i}{m_i} v_z^f)$$
$$e^{\frac{c_i}{m_i}t} v_{i,z} = \frac{c_i}{m_i} e^{\frac{c_i}{m_i}t}(g_z + \frac{c_i}{m_i} v_z^f) + K$$

Where K is an unknown constant of integration. We now get $v_{i,z}$ term by itself on the left hand side.

$$v_{i,z}(t) = \frac{c_i}{m_i}(g_z + \frac{c_i}{m_i} v_z^f) + Ke^{-\frac{c_i}{m_i}t}$$

Plugging in our boundary conditions to solve for K we get

$$K = v_{i,0,z} - (g_z + \frac{c_i}{m_i} v_z^f)\frac{m_i}{c_i}$$

Substituting back our value of K into our equation we get

$$v_{i,z}(t) = (\frac{g_z m_i}{c_i} + v_z^f) + (v_{i,0,z} - \frac{g_z m_i}{c_i} - v_z^f)e^{\frac{-c_i}{m_i}t} \tag{4.5}$$

Recall $m_i = \frac{4}{3}\pi R_i^3 \rho_i$ where ρ_i is the density of the particle. Substituting values of m_i and c_i and simplifying we get

$$v_{i,z}(t) = \frac{2 g_z R_i^2 \rho_i}{9\mu_f} + v_z^f + (v_{i,0,z} - \frac{2 g_z R_i^2 \rho_i}{9\mu_f} - v_z^f)e^{-\frac{9\mu_f}{2R_i^2\rho_i}t}$$

PART B SOLUTION:

As $t \to \infty$, $v_{i,z} = \frac{2 g_z R_i^2 \rho_i}{9\mu_f} + v_z^f$

Then as $R_i \to 0$, $v_{i,z} = v_z^f$. This implies small particles attain ambient velocities.

PART C SOLUTION:

$$\begin{aligned} r_{i,z}(t) &= r_{i,0,z} + v_{i,z}(t)t \\ &= \int v_{i,z}(t)dt \\ &= \int \frac{2 g_z R_i^2 \rho_i}{9\mu_f} + v_z^f + (v_{i,0,z} - \frac{2 g_z R_i^2 \rho_i}{9\mu_f} - v_z^f)e^{-\frac{9\mu_f}{2R_i^2\rho_i}t}dt \\ &= (\frac{2 g_z R_i^2 \rho_i}{9\mu_f} + v_z^f)t - (v_{i,0,z} - \frac{2 g_z R_i^2 \rho_i}{9\mu_f} - v_z^f)\frac{2R_i^2\rho_i}{9\mu_f}e^{-\frac{9\mu_f}{2R_i^2\rho_i}t} + K \end{aligned}$$

Solving for K:

$$r_{i,z}(0) = r_{i,0,z} = -(v_{i,0,z} - \frac{2 g_z R_i^2 \rho_i}{9\mu_f} - v_z^f)\frac{2R_i^2\rho_i}{9\mu_f} + K$$
$$K = r_{i,0,z} + (v_{i,0,z} - \frac{2 g_z R_i^2 \rho_i}{9\mu_f} - v_z^f)\frac{2R_i^2\rho_i}{9\mu_f}$$

Plugging in K into our equation we finally get:

$$r_{i,z}(t) = (\frac{2 g_z R_i^2 \rho_i}{9\mu_f} + v_z^f)t + (v_{i,0,z} - \frac{2 g_z R_i^2 \rho_i}{9\mu_f} - v_z^f)\frac{2R_i^2\rho_i}{9\mu_f}(1 - e^{-\frac{9\mu_f}{2R_i^2\rho_i}t}) + r_{i,0,z}$$

PART B SOLUTION:

If $v_z^f = 0$ as $t \to \infty$

$$r_{i,z} = r_{i,0,z} + \frac{2v_{i,z,0}R_i^2}{9\mu}$$

As $R_i \to 0$, $r_{i,z} = r_{i,0,z}$. This implies small particles travel shorter distances and are in the air longer.

5 References

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